STUDENT NAME:_

Mrs Gibson
Mr Keanan-Brown

2001
TRIAL HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

Total marks (**120**)

- Attempt Questions 1 – 8
- All questions are of equal value

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using blue or black pen
- A table of standard integrals is provided on page 13
- All necessary working should be shown in every question
- Marks may be deducted for careless or untidy work

QUESTION 1 Use a SEPARATE writing booklet **Marks**

(a) Find $\int \frac{\log x}{x}\,dx$ **1**

(b) Evaluate $\int_0^1 t e^{-t}\,dt$ **3**

(c) (i) Find the real numbers a, b and c such that

$$\frac{1}{x(1+x^2)} \equiv \frac{a}{x} + \frac{bx+c}{1+x^2}$$ **2**

(ii) Find $\int \frac{dx}{x(1+x^2)}$ **1**

(d) Evaluate $\int_0^4 \frac{x}{\sqrt{x+4}}\,dx$ **3**

(e) (i) If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$, show that for $n > 1$,

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)\,I_{n-2}$$ **3**

(ii) Hence find the area of the finite region bounded by

the curve $y = x^4 \cos x$ and the x axis for $0 \le x \le \frac{\pi}{2}$. **2**

QUESTION 2 Use a SEPARATE writing booklet **Marks**

(a) Express $\sqrt{3} - i$ in the form $r(\cos\theta + i\sin\theta)$. **3**

Hence show that $(\sqrt{3} - i)^9$ can be written as ci,
where c is real, and state the value of c.

(b) On the same Argand diagram, sketch the locus of a complex number Z if

(i) $|Z| = |Z - 4|$ **1**

(ii) $\arg(Z - i) = \dfrac{\pi}{4}$ **1**

Hence find the complex number $a + ib$ which satisfies both conditions simultaneously. **1**

(c) The complex number ω is given by $\omega = \dfrac{3 + i}{2 - i}$.

(i) Show that $\arg \omega = \dfrac{\pi}{4}$ and find $|\omega|$. **3**

(ii) Mark on an Argand diagram the points P and Q representing the complex numbers ω and ω^3 respectively. **1**

(iii) Show that Q can also represent the complex number $ki\omega$, and state the value of k. **1**

(d) (i) If $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$. **1**

(ii) By showing that the equation $3z^4 - z^3 + 2z^2 - z + 3 = 0$ can be written as $3(z^2 + z^{-2}) - (z + z^{-1}) + 2 = 0$, solve the equation $3z^4 - z^3 + 2z^2 - z + 3 = 0$, given that no root is real. **3**

QUESTION 3 Use a SEPARATE writing booklet **Marks**

(a) The ellipse, E, has equation $9x^2 + 25y^2 = 225$.

P is any point on the ellipse and A and B are the points $(5, 0)$ and $(-5, 0)$ respectively. AP, produced if necessary, meets the y axis in Q, and BP, also produced if necessary, meets the y axis in R.

The tangent at P meets the y axis in T.

(i) Find the eccentricity. **1**

(ii) Sketch the ellipse, E, showing the coordinates of its foci. **1**

(iii) Derive, in parametric form, the equation of the tangent at P. **2**

(iv) Prove that T is the midpoint of QR. **3**

(b) (i) Sketch, without the use of calculus, on the same set of coordinate axes and over the domain $-2\pi \le x \le 2\pi$, the three functions:

$$f(x) = \sin x \quad , \quad g(x) = \sin^2 x \quad \text{and} \quad h(x) = f(x) + g(x)$$ **4**

(ii) On separate coordinate axes sketch $y = \ln(h)$. **2**

(iii) Use an appropriate sketch from parts (i) or (ii) above to determine the number of real solutions of $e^x = \sin x + \sin^2 x$ over the domain $-2\pi \le x \le 2\pi$. **2**

QUESTION 4 Use a SEPARATE writing booklet **Marks**

(a) Solve, graphically or otherwise, $\dfrac{1}{x-1} < |3x-5|$ **4**

(b) The area enclosed by the curves $y = \dfrac{1}{x^2}$, $y = \dfrac{1}{x}$ *and* $x = \dfrac{1}{10}$ **5**

is rotated about the y axis.

Use the method of cylindrical shells to find the volume of the solid formed.

(c) (i) Use the formula $A = \dfrac{1}{2} a b \sin C$ to show that the area of a regular pentagon of side D units is given by

$$A = \frac{5}{2} D^2 \frac{\sin^2 54^\circ}{\sin 72^\circ}$$ **3**

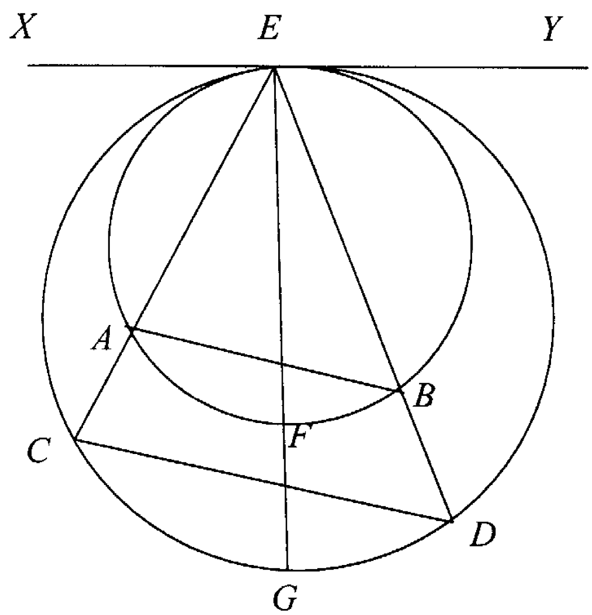
(ii) The area enclosed by $y = x^2$ and $y = 3$ is the base of a certain solid. Cross-sections of the solid, parallel to the x axis are regular pentagons with one side on the base.

Find the volume of the solid. **3**

QUESTION 5 Use a SEPARATE writing booklet **Marks**

(a)

Not to scale



Two circles, with diameters 2 cm and 3 cm, touch internally at E.
The tangent XY is drawn through their common point of contact.
The line of centres is drawn, cutting the smaller circle in F and the larger circle in G.
Chord EA in the smaller circle is produced to meet the larger circle in C. Similarly chord EB is produced to meet the larger circle in D.
Points A, B are joined and points C, D are joined.

Copy (or trace) the circles into your answer booklet.

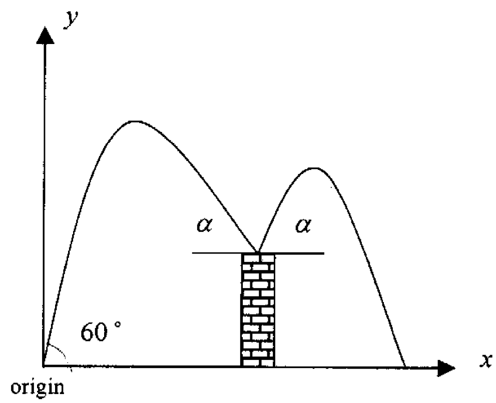
(i) Prove that AB is parallel to CD. **2**

(ii) Given that AB = 1.8 cm, prove that CD = 2.7 cm. **3**

QUESTION 5 CONTINUES ON THE NEXT PAGE

 Marks

(b)



In this question, neglect air resistance and assume that acceleration due to gravity $g = 10 \ m s^{-2}$.

A ball is throw with initial speed $8 \ m s^{-1}$ at an angle of 60^o to the horizontal. The ball lands on the top of a brick wall, and it bounces off again.

(i) Derive all the equations of motion of the ball. **2**

(ii) If the ball hits the wall after $\frac{\sqrt{27}}{5}$ seconds, show that the wall is 1.8 metres high and $12\frac{\sqrt{3}}{5}$ metres horizontally from the origin. **2**

(iii) Also show that the ball strikes the top of the wall at an angle α, where $\tan\alpha = \frac{\sqrt{3}}{2}$. **2**

Now assume that the ball bounces forward off the wall with the same angle α and assume that the speed of projection off the wall is equal to the speed of impact onto the wall.

(iv) Show that the speed of projection off the wall is $2\sqrt{7}$ m/s. **1**

(v) Write down a new set of equations of motion of the ball from the moment it bounces off the wall. **2**

(vi) Calculate where the ball first reaches the ground measured from the original point of projection. **1**

QUESTION 6 Use a SEPARATE writing booklet **Marks**

(a) Consider two polynomials $P(x)$ and $F(x)$.
When $P(x)$ is divided by x^2+6x+8 the remainder is $2x-11$.
When $F(x)$ is divided by x^2+6x+8 the remainder is $x+4$.
With each division the quotient is the same.

(i) Show that $P(x)$ and $F(x)$ must have the same degree. **1**

(ii) Write an expression for $P(x) - F(x)$. **1**

(iii) Find the remainder when $P(x)$ is divided by $x+4$. **1**

(b) Consider any equation of the form $x^3+mx^2-n=0$ where $m,n \neq 0$.

(i) Prove that such an equation cannot have a triple root. **2**

(ii) Assuming that the equation has a double root, find the relation between m and n. **2**

QUESTION 6 CONTINUES ON THE NEXT PAGE

(c) Ice cream begins to melt at 0° C. When a frozen ice cream thaws, its temperature slowly rises until it reaches 0° C. An ice cream manufacturer has developed a new wrapper that slows down the rate of thawing after an ice cream is taken from the shop freezer. Ice creams are stored at -20° C in the shop freezer. Temperature measurements recorded show that it takes 400 seconds for the temperature of an ice cream to rise from -20° C to -4° C.

(i) Assuming a constant rate of thawing, estimate when the temperature of the ice cream will reach 0° C. **2**

In fact, the rate of thawing of the ice cream is *not* constant and the temperature, $\theta\ ^\circ$C, of the ice cream, t seconds after being taken out of the shop freezer is given by $\ln(60 - \theta) = c - kt$, where k and c are constants.

(ii) Find $\dfrac{d\theta}{dt}$ and hence show that the rate of thawing is proportional to the amount by which the temperature is below 60° C. **3**

(iii) Find the values of c and k and hence find an improved estimate of the time before the ice cream begins to melt. **3**

QUESTION 7 Use a SEPARATE writing booklet **Marks**

(a) Prove, by mathematical induction, that

$$\sum_{r=1}^{n} \cos(2r-1)\theta = \frac{\sin n\theta \cos n\theta}{\sin\theta}, \qquad \sin\theta \neq 0. \qquad \mathbf{5}$$

(b) Jennifer dropped her lightweight mobile phone off the top of the library building. The phone has a mass of 12 *grams*. Take acceleration due to gravity $g = 10\ ms^{-2}$, and air resistance proportional to velocity where the constant of proportionality is $\frac{3}{10}$.

(i) Show that the equation of motion of the phone is given by $\ddot{x} = 10 - 25v$, where $v\ ms^{-1}$ is the velocity of the phone after t seconds. **1**

Determine

(ii) $v(t)$. i.e. an expression for velocity in terms of time. **3**

(iii) v_T i.e. the value of its terminal velocity in ms^{-1}. **1**

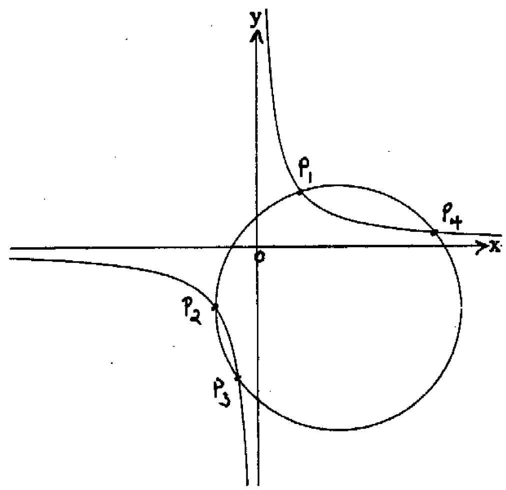
(iv) the time elapsed before the velocity $v(t)$ has reached 99% of v_T, giving your answer correct to 2 decimal places. **2**

(v) $x(t)$ i.e. an expression for the distance fallen in terms of time t. **2**

(vi) Show that if Jennifer had reacted quickly, she could have caught her mobile phone if she had reached down 38 cm within 1 second of dropping her phone. **1**

QUESTION 8 Use a SEPARATE writing booklet **Marks**

(a)



The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the rectangular hyperbola $x = kt,\ y = \frac{k}{t}$ in four distinct points P_i with parameters t_i respectively, as shown in the sketch above.

(i) Prove that $t_1 t_2 t_3 t_4 = 1$. **2**

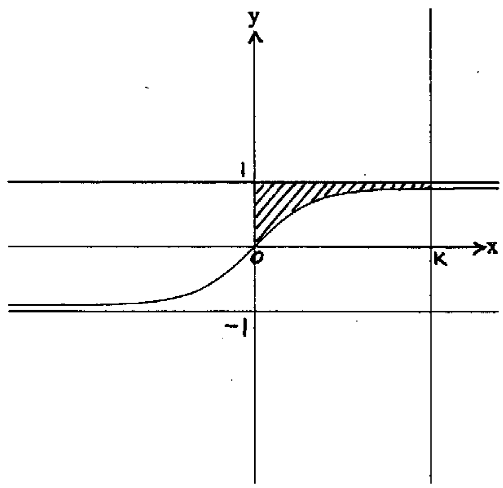
(ii) Show that the chord joining points P_1 and P_2 has equation

$$x + t_1 t_2 y = k(t_1 + t_2).$$ **2**

(iii) If $P_1 P_2$ passes through the origin, prove that $P_3 P_4$ is a diameter of the circle. **4**

QUESTION 8 CONTINUES ON THE NEXT PAGE

(b)



The sketch above shows the curve with equation $y = \dfrac{e^{2x} - 1}{e^{2x} + 1}$,
its asymptotes $y = \pm 1$, and the line $x = K$, where $K > 0$.

(i) Using the substitution $u = e^{2x}$, or otherwise, show that the area in the first quadrant bounded by the curve $y = \dfrac{e^{2x} - 1}{e^{2x} + 1}$ and the lines $x = 0$, $y = 1$ and $x = K$ (that is, the shaded area in the sketch above) is given by $A = \log 2 + 2K - \log(e^{2K} + 1)$. **5**

(ii) Explain why this area is always less than $\log 2$, no matter how large K may be. **2**

END OF PAPER

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Question 1

a) $\int \frac{\ln x}{x}\,dx = \int \ln x \; d\ln x$

$= \frac{(\ln x)^2}{2} + C$

b) $\int_0^1 t e^{-t}\,dt = \int_0^1 -t\; de^{-t}$

$= -\left[t e^{-t}\right]_0^1 + \int_0^1 e^{-t}\, d(t)$

$= -\frac{1}{e} + \left[-e^{-t}\right]_1^0$

$= -\frac{1}{e} + 1 - \frac{1}{e}$

$= 1 - \frac{2}{e}$

alternative method

Let $u = t$, $\frac{dv}{dt} = e^{-t}$

$\frac{du}{dt} = 1$, $v = -e^{-t}$

c) (i) $\frac{1}{x(1+x^2)} = \frac{a}{x} + \frac{bx+c}{1+x^2}$

$1 = a(1+x^2) + bx^2 + cx$

$a + b = 0$

$c = 0$

$a = 1$

$\therefore b = -1$

$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$

(ii) $\int \frac{dx}{x(1+x^2)} = \ln|x| - \frac{1}{2}\ln(1+x^2) + C, \; x \neq 0$

d). $\int_0^4 \frac{x}{\sqrt{x+4}}\,dx = \int_0^4 \frac{x+4 \quad -4}{\sqrt{x+4}}\,dx$

$= \int_0^4 (x+4)^{\frac{1}{2}}\,dx - \frac{4}{\sqrt{x+4}}\,dx$

$= \left[\frac{2}{3}(x+4)^{\frac{3}{2}}\right]_0^4 - \left[\frac{4(x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^4$

$= \frac{2}{3}\left(8^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) - 8\left(8^{\frac{1}{2}} - 4^{\frac{1}{2}}\right)$

$= \frac{16}{3}(2 - \sqrt{2})$

alternative method

let $u = x+4$

$du = dx$

when $x = 0, u = 4$

$x = 4, u = 8$

$I = \int_4^8 \frac{u-4}{\sqrt{u}}\,du$

$= \int_4^8 \sqrt{u} - \frac{4}{\sqrt{u}}\,du$

$= \left[\frac{2}{3}u^{3/2} - 8u^{\frac{1}{2}}\right]_4^8$

$= \left[\frac{2}{3}(2\sqrt{2})^3 - 8(2\sqrt{2})\right] - \left[\frac{2}{3}(2)^3 - 8(2)\right]$

$= \frac{16}{3}(2 - \sqrt{2})$

Question 1 (e)

(i) $I_n = \int_0^{\pi/2} x^n \cos x\,dx$

$= \int_0^{\pi/2} x^n\, d\sin x$

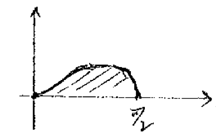
$= \left[x^n \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \cdot n x^{n-1}\,dx$

$= \left(\frac{\pi}{2}\right)^n + n\int_0^{\pi/2} x^{n-1}\, d\cos x$

$= \left(\frac{\pi}{2}\right)^n + n\underbrace{\left[x^{n-1}\cos x\right]_0^{\pi/2}}_{0} - n\int_0^{\pi/2} \cos x \cdot (n-1)x^{n-2}\,dx$

$= \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$

(ii)



$A = I_4$

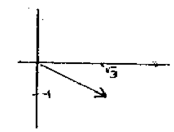
$= \left(\frac{\pi}{2}\right)^4 - 4\times 3\left[\left(\frac{\pi}{2}\right)^2 - 2\int_0^{\pi/2} \cos x\,dx\right]$

$= \left(\frac{\pi}{2}\right)^4 - 3\pi^2 + 24\left[\sin x\right]_0^{\pi/2}$

$= \left(\frac{\pi}{2}\right)^4 - 3\pi^2 + 24.$

Question 2 (a)

$\sqrt{3} - i = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$



$(\sqrt{3} - i)^9 = 2^9\,\text{cis}\left(-\frac{9\pi}{6}\right)$

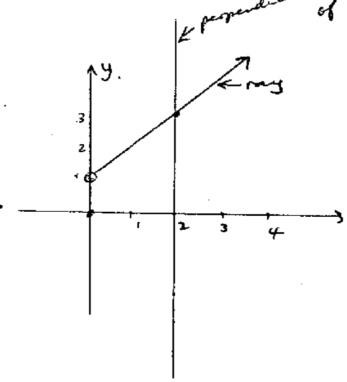
$= 512\,\text{cis}\; -\frac{3\pi}{2}$

$= 512\,\text{cis}\,\frac{\pi}{2}$

$= 512\,i$

$= ci$ where $c = 512$.

b) $a+ib$ that satisfies simultaneously is $(2+3i)$.



Question 2 (c)

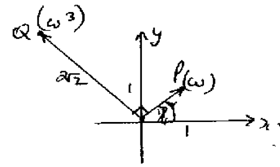
(i) $\omega = \frac{3+i}{2-i} \times \frac{2+i}{2+i}$

$= \frac{5+5i}{5}$

$= 1+i$

$\arg \omega = \pi/4$

$|\omega| = \sqrt{2}$.



(ii) see opposite ↗

(iii) $ki\omega$ represents ω^3 where $k=2$

(because the vector is longer by a factor of 2, and it is rotated anticlockwise $90°$ hence multiplied by i)

(d)(i) $z^n = \cos n\theta + i \sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$; but $\cos(-n\theta) = \cos n\theta$ and $\sin(-n\theta) = -\sin n\theta$

$z^n + z^{-n} = 2\cos n\theta$ *

(ii) $3z^4 - z^3 + 2z^2 - z + 3 = 0$

divide by z^2 for $z \neq 0$

$3z^2 - z + 2 - \frac{1}{z} + \frac{3}{z^2} = 0$

$3\left(z^2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) + 2 = 0$

$3.(2\cos 2\theta) - (2\cos\theta) + 2 = 0$ (using the result above *)

$3\cos 2\theta - \cos\theta + 1 = 0$

$3(2\cos^2\theta - 1) - \cos\theta + 1 = 0$

$6\cos^2\theta - \cos\theta - 2 = 0$

$(3\cos\theta - 2)(2\cos\theta + 1) = 0$

$\cos\theta = \frac{2}{3}$, $\cos\theta = -\frac{1}{2}$

$\sin\theta = \pm\frac{\sqrt{5}}{3}$, $\sin\theta = \pm\frac{\sqrt{3}}{2}$

$\therefore z = \frac{2 \pm i\sqrt{5}}{3}$ or $z = \frac{-1 \pm \sqrt{3}}{2}$

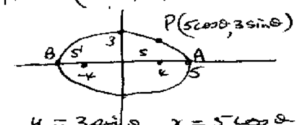
Question 3 (a)

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

$9 = 25(1-e^2)$

(i) $e = 4/5$

(ii) foci $(\pm 4, 0)$



$y = 3\sin\theta, \quad x = 5\cos\theta$

(iii) $\therefore \frac{dy}{dx} = -\frac{3\cos\theta}{5\sin\theta}$

Equation of tangent at P

$\therefore y - 3\sin\theta = \frac{-3\cos\theta}{5\sin\theta}(x - 5\cos\theta)$ ①

(iv) Coords of T: from ①

$\left(0, \frac{3}{\sin\theta}\right)$

Equation of AP gradient $= -\frac{3\sin\theta}{5-5\cos\theta}$

$y = -\frac{3\sin\theta}{5(1-\cos\theta)}(x-5)$ ②

$\therefore$ Coords of Q: from ②

$\left(0, \frac{3\sin\theta}{1-\cos\theta}\right)$

Equation of BP gradient $= \frac{3\sin\theta}{5+5\cos\theta}$

$y = \frac{3\sin\theta}{5(1+\cos\theta)}(x+5)$ ③

$\therefore$ Coords of R: from ③

$\left(0, \frac{3\sin\theta}{1+\cos\theta}\right)$

Midpt of QR

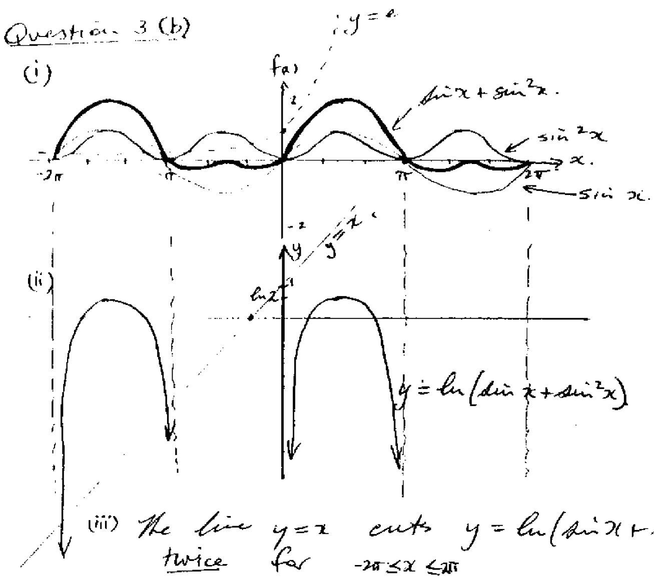
$\left(0, \frac{1}{2}\frac{3\sin\theta(1+\cos\theta+1-\cos\theta)}{1-\cos^2\theta}\right)$

$= \left(0, \frac{3}{\sin\theta}\right)$

= coords of T as required.

Question 3 (b)

(i)

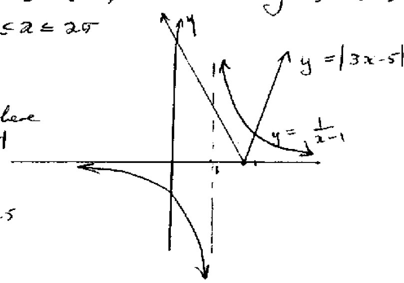


(iii) The line $y = x$ cuts $y = \ln(\sin x + \sin^2 x)$ twice for $-2\pi \le x \le 2\pi$

or

the curve $y = e^x$ cuts the curve $y = \sin x + \sin^2 x$ twice for $-2\pi \le x \le 2\pi$

Question 4 (a)



From the sketch opposite there is one solution of $\frac{1}{x-1} = |3x-5|$ that is for $\frac{1}{x-1} = 3x - 5$

$1 = 3x^2 - 8x + 5$

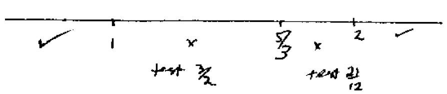
$3x^2 - 8x + 4 = 0$

$(3x-2)(x-2) = 0$

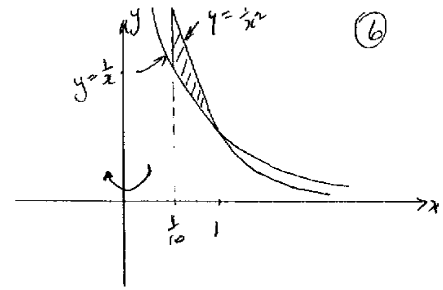
$x = \frac{2}{3}, x = 2$ the solution $(x > 1)$ is $x = 2$.

and so the solution of $\frac{1}{x-1} < |3x - 5|$ is $x < 1$, $x > 2$.

a critical values approach, testing



Question 4(b).



$V = 2\pi \int_{\frac{1}{10}}^{1} xy \, dx$

$= 2\pi \int_{\frac{1}{10}}^{1} x\left(\frac{1}{x^2} - \frac{1}{x}\right) dx$

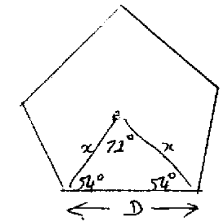
$= 2\pi \int_{\frac{1}{10}}^{1} \frac{1}{x} - 1 \, dx$

$= 2\pi \left[\ln x - x\right]_{\frac{1}{10}}^{1}$

$= 2\pi \left\{(-1) - \left(\ln \frac{1}{10} - \frac{1}{10}\right)\right\}$

$= 2\pi \left(\ln 10 - \frac{9}{10}\right) \ u^3$.

Question 4 (c)



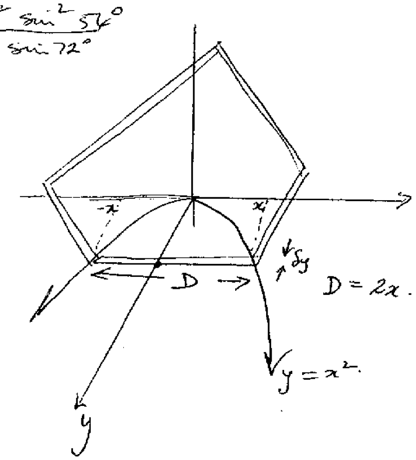
(i) Area of pentagon

$= 5 \times \frac{1}{2} \times x^2 \sin 72$.

Now $\frac{x}{\sin 54} = \frac{D}{\sin 72}$.

$\therefore$ $\text{Area}_{\text{pentagon}} = \frac{5}{2} \cdot \frac{D^2 \sin^2 54°}{\sin 72°}$

(ii)



$\delta V = \frac{5}{2} (2x)^2 \frac{\sin^2 54°}{\sin 72°} \delta y$

$= 10 \, y \, \frac{\sin^2 54°}{\sin 72°} \delta y$

$V = 10 . \frac{\sin^2 54°}{\sin 72°} \int_0^3 y \, dy$

$= 5 \frac{\sin^2 54°}{\sin 72°} \left[y^2\right]_0^3$

$= 45 \frac{\sin^2 54°}{\sin 72°} \ u^3$.

/)

Question 5 (a)

in △ EAB, △ ECD

① { $Y\hat{E}B = E\hat{A}B$ (angle between chord and tangent = angle in alternate segment)
$Y\hat{E}B = E\hat{C}D$ (" ").

$A\hat{E}B = C\hat{E}D$ (common angle)

∴ △ EAB ||| △ ECD (equiangular)

∴ $\frac{EB}{ED} = \frac{AB}{CD}$ ② (corresponding sides of similar triangles)

Join FB, GD

In △ EFB, △ EGD.

$E\hat{B}F = 90°$ (angle in a semicircle)

$E\hat{D}G = 90°$ (" ")

$B\hat{E}F = \hat{D}EG$ (common angle)

∴ △ EFB ||| △ EGD (equiangular)

∴ $\frac{EB}{ED} = \frac{EF}{EG}$ ③ (corresponding sides of similar △s)

(i) from ① AB ∥ CD (($E\hat{A}B = E\hat{C}D$ proved above); corresponding angles equal)

(ii) from ② + ③.

$\frac{AB}{CD} = \frac{EF}{EG}$

$= \frac{2}{3}$ (given)

∴ Since AB = 1·8, $CD = 1·8 \times \frac{3}{2}$

ie CD = 2·7 cm.

Question 5(b)

(i) $\ddot{x} = 0$; $t=0, \dot{x}=4$ $\quad$ $\ddot{y} = -10$; $t=0$, $\dot{y} = 4\sqrt{3}$

∴ $\dot{x} = 4$; $t=0, x=0$ $\quad$ ∴ $\dot{y} = -10t + 4\sqrt{3}$; $t=0$, $y=0$

∴ $x = 4t + 0$ $\quad$ ∴ $y = -5t^2 + 4t\sqrt{3} + 0$.

(ii) $x = 4t$
$= 4\frac{\sqrt{27}}{5}$
$= \frac{12\sqrt{3}}{5}$

$y = -5\left(\frac{\sqrt{27}}{5}\right)^2 + 4\sqrt{3} \cdot \frac{\sqrt{27}}{5}$
$= \frac{9}{5}$

(iii) When $t = \frac{\sqrt{27}}{5}$ $\quad$ $\dot{x} = 4$ $\quad$ $\dot{y} = 2\sqrt{3}$

$\tan \alpha = \frac{2\sqrt{3}}{4}$
$= \frac{\sqrt{3}}{2}$

(iv) $\dot{x} = 4$ $\quad$ $\dot{y} = 2\sqrt{3}$

Velocity of projection $= \sqrt{16+12}$
$= 2\sqrt{7}$.

Question 5(b) continued

(v) $\ddot{x} = 0$ $\quad$ $\ddot{y} = -10$

$\dot{x} = 2\sqrt{7} \cos\alpha$ $\quad$ $\dot{y} = -10t + \frac{\sqrt{3}}{\sqrt{7}} \cdot 2\sqrt{7}$

$= 4$ as before $\quad$ $= -10t + 2\sqrt{3}$.

$x = 4t + \frac{12\sqrt{3}}{5}$ ② $\quad$ $y = -5t^2 + 2\sqrt{3} + 1·8$ ①

(vi) when $y = 0$ in equation ① above

$t = \frac{-2\sqrt{3} \pm \sqrt{12+36}}{-10}$

$= \frac{-6\sqrt{3}}{-10}$ since $t > 0$

$= \frac{3\sqrt{3}}{5}$

from ② $x\left(\frac{3\sqrt{3}}{5}\right) = \frac{12\sqrt{3}}{5} + \frac{12\sqrt{3}}{5}$

$= \frac{24\sqrt{3}}{5}$ m from the original point of projection

Question 6(a)

(i) $P(x) = (x^2+6x+8) \cdot Q(x) + (2x-11)$ ① (same quotient)

$F(x) = (x^2+6x+8) \cdot Q(x) + (x+4)$ ②

deg $P(x) = 2 +$ deg Q

deg $F(x) = 2 +$ deg Q $\quad$ ∴ deg P = deg F.

(ii) $P(x) - F(x) = (2x-11) - (x+4)$ from ① & ② above

$= x - 15$

(iii) $P(-4) = (16 - 24 + 8) \cdot Q(x) + (2 \times -4 - 11)$

$= 0 - 19$

$= -19$

Question 6(b)

(i) $P(x) = x^3 + mx^2 - n$

$P'(x) = 3x^2 + 2mx$

$P''(x) = 6x + 2m$

$= 0$ for $x = -\frac{m}{3}$

now $P'\left(-\frac{m}{3}\right) = \frac{m^2}{3} - \frac{2m^2}{3}$

$\neq 0$ since $m \neq 0$ given

∴ no triple root.

(ii) $P'(x) = 0$ for $x(3x+2m) = 0$

$x = 0$ or $x = -\frac{2m}{3}$

but $P(0) \neq 0$ since $n \neq 0$ given

Sub $x = -\frac{2m}{3}$ into $P(x)$ ⇒

$-\frac{8m^3}{27} + \frac{4m^3}{9} - n = 0$

$n = \frac{4m^3}{27}$

alternative method, Q 6(b)

(i) let roots be α, α, α.

then $3\alpha = -m$ ①

$3\alpha^2 = 0$ ②

$\alpha^3 = n$ ③

from ② $\alpha^2 = 0$ ∴ $\alpha^3 = 0$

but $n \neq 0$ given ∴ no triple root

(ii) let roots be α, α, β

then $2\alpha + \beta = -m$ ①

$\alpha^2 + 2\alpha\beta = 0$ ②

$\alpha^2\beta = n$ ③

from ② $\alpha(\alpha + 2\beta) = 0$ $\quad$ $\alpha = 0$ or $\alpha + 2\beta = 0$

but $P(0) \neq 0$ ∵ $n \neq 0$

∴ $\alpha \neq 0$

ie $\alpha + 2\beta = 0$

$2\alpha + 4\beta = 0$ ④

$3\beta = m$

$\beta = m/3$

$\alpha = -2\beta$

$\alpha^2 = 4m^2/9$

∴ $\frac{4m^2}{9} \cdot \frac{m}{3} = n$ ∴ $4m^3 = 27n$.

(9)

Question 6(e)

(i) -20° to -4° = rise of 16° in 400 sec

i.e. rise 1° per 25 sec

$\therefore$ from -20° to 0° will take 20×25 sec

= 500 sec.

Alternately

$\frac{dT}{dt} = k$, k a constant.

$T = kt + c$ but $T(0) = -20$

$\therefore T = kt - 20$ and $T(400) = -4$

$\therefore -4 = 400k - 20$ $\therefore k = \frac{1}{25}$

$\therefore$ When $T = 0^\circ$ $0 = \frac{t}{25} - 20$

$\therefore t = 500$ sec.

(ii) $\ln(60-\theta) = c - kt$ (θ is temperature)

$60 - \theta = e^{c-kt}$ —①

$\theta = 60 - e^{c-kt}$

$\frac{d\theta}{dt} = k e^{c-kt}$

$= k(60-\theta)$ from ①

$\propto (60-\theta)$ since k is a constant.

(iii) from $\ln(60-\theta) = c - kt$

when $t = 0$ $\theta = -20$

$\therefore$ $\boxed{\ln 80 = c}$

and when $t = 400$, $\theta = -4$

$\therefore \ln 64 = \ln 80 - 400k$

$400k = \ln \frac{80}{64}$

$\boxed{k = \frac{1}{400}\ln\left(\frac{5}{4}\right)}$.

Now thawing ends and melting begins at 0°

$\therefore$ when $\theta = 0^\circ$

$\ln 60 = \ln 80 - \frac{1}{400}\ln\left(\frac{5}{4}\right) \cdot t$

$\frac{1}{400}\ln\left(\frac{5}{4}\right) t = \ln\frac{80}{60}$

$t = 400 \cdot \ln\left(\frac{4}{3}\right) \div \ln\left(\frac{5}{4}\right)$

$= 515.68\ldots$

≈ 516 seconds.

Question 7(a)

When $n = 1$ LHS = $\cos\theta$, RHS = $\cos\theta$

$\therefore$ result is true for $n = 1$

consider $n = k$, $k \geq 1$ an integer

assume $\cos\theta + \cos 3\theta + \ldots + \cos(2k-1)\theta = \sin k\theta . \cos k\theta / \sin\theta$

To show $\cos\theta + \cos 3\theta + \cos 5\theta + \ldots + \cos(2k-1)\theta + \cos(2k+1)\theta$

$= \frac{\sin(k+1)\theta \cos(k+1)\theta}{\sin\theta}$.

Now $\cos\theta + \cos 3\theta + 5\theta + \ldots + \cos(2k+1)\theta = \frac{\sin k\theta}{\sin\theta}\cos k\theta + \cos(2k+1)\theta \times \frac{\sin\theta}{\sin\theta}$

$= \frac{\sin k\theta \cos k\theta + \cos(2k+1)\theta \sin\theta}{\sin\theta}$ ①

But $\sin(2k+2)\theta = \sin(2k+1)\theta \cos\theta + \cos(2k+1)\theta \sin\theta$

$\therefore \cos(2k+1)\theta \sin\theta = \sin(2k+2)\theta - \sin(2k+1)\theta\cos\theta$

And line ① becomes: $\frac{\sin k\theta \cos k\theta + (\sin(2k+2)\theta - \sin(2k+1)\theta\cos\theta)}{\sin\theta}$

$= \frac{\sin k\theta\cos k\theta + \sin(2k+2)\theta - (\sin 2k\theta\cos\theta + \cos 2k\theta\sin\theta)\cos\theta}{\sin\theta}$

$= \frac{\sin k\theta\cos k\theta + \sin(2k+2)\theta - (\sin 2k\theta\cos^2\theta + \cos 2k\theta\sin\theta\cos\theta)}{\sin\theta}$

$= \frac{\frac{1}{2}(\sin 2k\theta) + \sin(2k+2)\theta - (\sin 2k\theta(\frac{1}{2}(1+\cos 2\theta)) + \cos 2k\theta \cdot \frac{1}{2}\sin 2\theta)}{\sin\theta}$

$= \frac{\sin(2k+2)\theta}{\sin\theta} - \frac{1}{2}\frac{\sin(2k+2)\theta}{\sin\theta}$

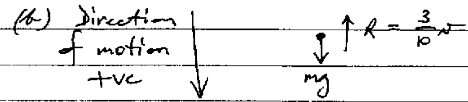
$= \frac{1}{2}\frac{\sin 2(k+1)\theta}{\sin\theta}$

$= \frac{\sin(k+1)\theta . \cos(k+1)\theta}{\sin\theta}$ as required

$\therefore$ If result is true for $n = k$, then it is true for $n = k+1$

But the result is true for $n = 1$, so it is true for $n = 1+1$ $= 2$ and so on for all positive integers n.

(b)



(i) Taking origin at top of library,

$$m\ddot{x} = mg - \frac{3}{10}v$$

$$\ddot{x} = g - \frac{3}{10m}v$$

Now, $m = 12$ grams $= 0.012$ kg

$$\therefore \ddot{x} = g - \frac{3}{10 \times 0.012}v \quad \boxed{1}$$

$$\therefore \ddot{x} = 10 - 25v$$

(ii)

$$\frac{dv}{dt} = 10 - 25v$$

$$\frac{dt}{dv} = \frac{1}{10-25v}$$

$$t = \int_0^v \frac{1}{10-25v}\,dv$$

$$= -\frac{1}{25}\Big[\ln(10-25v)\Big]_0^v$$

$$= -\frac{1}{25}\Big[\ln(10-25v) - \ln 10\Big]$$

$$= -\frac{1}{25}\ln\left(\frac{10-25v}{10}\right)$$

$$\therefore -25t = \ln\left(\frac{10-25v}{10}\right)$$

$$\frac{10-25v}{10} = e^{-25t}$$

$$10 - 25v = 10e^{-25t}$$

$$25v = 10 - 10e^{-25t} \quad \boxed{3}$$

$$\therefore v = \frac{10}{25}\left(1 - e^{-25t}\right)$$

$$\therefore v(t) = \frac{2}{5}\left(1 - e^{-25t}\right)$$

(iii) As $t \to \infty$, $e^{-25t} \to 0$ $\boxed{1}$

$\therefore$ Terminal velocity $v_T = \frac{2}{5}$

(iv) When velocity has reached 99% of v_T, $(= 0.396)$

then $t = \int_0^{0.396} \frac{1}{10-25v}\,dv$

$$= -\frac{1}{25}\Big[\ln(10-25v)\Big]_0^{0.396}$$

$$= -\frac{1}{25}\left[\ln\left(\frac{10-25\times 0.396}{10}\right)\right] \quad \boxed{2}$$

$$= 0.184\ldots$$

$= 0.18$ seconds (to 2 d.p.)

(v) $v(t) = \frac{2}{5}\left(1 - e^{-25t}\right)$

$$\frac{dx}{dt} = \frac{2}{5} - \frac{2}{5}e^{-25t}$$

$$\therefore x = \int_0^t \left(\frac{2}{5} - \frac{2}{5}e^{-25t}\right) dt$$ [2]

$$= \left[\frac{2}{5}t + \frac{2}{125}e^{-25t}\right]_0^t$$

$$\therefore x(t) = \frac{2}{5}t + \frac{2}{125}e^{-25t} - \frac{2}{125}$$

(vi) When $t = 1$,

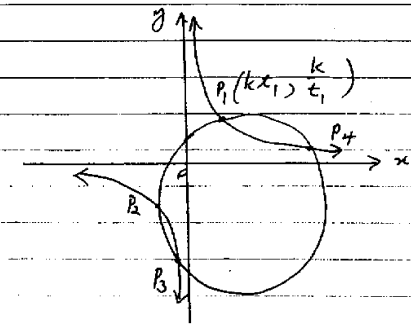
$$x = \frac{2}{5} + \frac{2}{125}e^{-25} - \frac{2}{125}$$ [1]

$$= 0.384 \text{ m}$$

$$\doteqdot 38 \text{ cm}$$

Question 8

(a)



(i) As points $(kt, \frac{k}{t})$ lie on circle, coordinates satisfy equation

$$\therefore (kt)^2 + \left(\frac{k}{t}\right)^2 + 2g(kt) + 2f\left(\frac{k}{t}\right) + c = 0$$

$$k^2t^2 + \frac{k^2}{t^2} + 2gkt + \frac{2fk}{t} + c = 0$$

$$k^2t^4 + 2gkt^3 + 2fkt + ct^2 + k^2 = 0$$ [2]

Roots of this equation are parameters t_1, t_2, t_3, t_4.

Product of roots $= t_1t_2t_3t_4 = \frac{k^2}{k^2}$

$$\therefore t_1t_2t_3t_4 = 1$$

(ii) Eqn of chord P_1P_2 is

$$\frac{y-\frac{k}{t_1}}{x-kt_1} = \frac{\frac{k}{t_2}-\frac{k}{t_1}}{kt_2-kt_1} = \frac{\frac{kt_1-kt_2}{t_1t_2}}{kt_2-kt_1}$$

$$\therefore \frac{y-\frac{k}{t_1}}{x-kt_1} = \frac{-1}{t_1t_2}$$

[2]

$$t_1t_2y - kt_2 = -x + kt_1$$

$$\therefore x + t_1t_2y = k(t_1+t_2)$$

(iii) If P_1P_2 passes through origin, coords. satisfy equation

$$\therefore 0 = k(t_1+t_2)$$

$$\therefore t_1 = -t_2$$

If P_3P_4 is a diameter of circle, then $P_1P_4 \perp P_1P_3$

$$\text{grad. of } P_1P_4 = \frac{\frac{k}{t_4}-\frac{k}{t_1}}{kt_4-kt_1}$$

$$= \frac{\frac{kt_1-kt_4}{t_4t_1}}{kt_4-kt_1}$$

$$= -\frac{1}{t_1t_4}$$

Similarly, grad. of $P_1P_3 = \frac{-1}{t_1t_3}$

Now, if $P_1P_4 \perp P_1P_3$, product of gradients $= -1$

$$\therefore \frac{-1}{t_1t_4} \cdot \frac{-1}{t_1t_3} = \frac{1}{t_1t_1t_3t_4}$$

$$= \frac{1}{t_1 \cdot -t_2t_3t_4} \text{, as } t_1 = -t_2$$

[4]

$$= \frac{-1}{t_1t_2t_3t_4}$$

$$= -1$$

$\therefore P_3P_4$ is diameter of circle

(b) Unshaded area $= \int_0^k \frac{e^{2x}-1}{e^{2x}+1}\,dx$

Let $u = e^{2x}$

$du = 2e^{2x}\,dx$

$\therefore dx = \frac{1}{2u}\,du$

When $x=0$, $u=1$

When $x=k$, $u=e^{2k}$

$\therefore$ Unshaded area $= \int_1^{e^{2k}} \frac{u-1}{u+1}\cdot\frac{1}{2u}\,du$

$= \frac{1}{2}\int_1^{e^{2k}} \frac{u-1}{u(u+1)}\,du$ 5

$= \frac{1}{2}\int_1^{e^{2k}} \frac{u}{u(u+1)} - \frac{1}{u(u+1)}\,du$

$= \frac{1}{2}\int_1^{e^{2k}} \left(\frac{1}{u+1} - \frac{1}{u} + \frac{1}{u+1}\right)du$

$= \frac{1}{2}\left[2\ln(u+1) - \ln u\right]_1^{e^{2k}}$

$= \frac{1}{2}\left[2\ln(e^{2k}+1) - \ln(e^{2k}) - 2\ln 2\right]$

$= \ln(e^{2k}+1) - k - \ln 2$

$\therefore$ Shaded area $= k\times 1 - \left[\ln(e^{2k}+1) - k - \ln 2\right]$

$= 2k - \ln(e^{2k}+1) + \ln 2$

(ii) Now, $e^{2k}+1 > e^{2k}$ for all $k>0$

$\therefore \ln(e^{2k}+1) > \ln e^{2k}$ 2

$\ln(e^{2k}+1) > 2k$

$\therefore 2k - \ln(e^{2k}+1) < 0$

So, area is always less than $\ln 2$